Electron Transport in the Presence of Magnetic Fields

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In the presence of a magnetic field, a two-dimensional conductor will experience various corrections to its resistance. A transverse conductivity, known as the Hall conductivity, will appear that creates a potential drop perpendicular to the direction of current flow. Furthermore, the longitudinal conductivity oscillates at a function of magnetic field strength for strong fields. The transverse conductivity is enhanced in a ferromagnetic material and exists even at zero magnetic field.

I. INTRODUCTION

The Hall effect is one of the oldest and most studied electromagnetic phenomena. The classical effect was first observed in 1879 by Edwin Hall[1], who found that passing a current through a gold leaf in the presence of a magnetic field resulted in the development of a transverse ohmic potential. Notably, this preceded the discovery of the electron by eighteen years.

In spite of its early discovery, the effect has theoretical and experimental features that require high precision and a deep understanding of the underlying physics. Here, two aspects are discussed. The first is the appearance of plateaus with very precise resistances in the Hall resistance at high magnetic fields, known as the integer quantum Hall effect. The second is the existence of a Hall effect in ferromagnetic materials greater than would be predicted based only on their internal magnetic field. This effect was first discovered in iron by Hall in 1881, but the mechanisms are still being debated.

II. CLASSICAL HALL EFFECT

In the presence of a magnetic field $\vec{B}$ in the $\hat{z}$ direction, electrons traveling through a conductor in the $\hat{x}$ direction will experience a force in the $\hat{y}$ direction:

$$\vec{F}_B = e\vec{v} \times \vec{B}. \tag{1}$$

The classical drift velocity is given by

$$\vec{v} = \frac{\vec{I}}{neA}, \tag{2}$$

which gives a magnetic force of

$$\vec{F}_B = \frac{\vec{I} \times \vec{B}}{nA}. \tag{3}$$

This results in an electric field that matches that force:

$$\vec{F}_E = Ee = \frac{\vec{I} \times \vec{B}}{nA} = \vec{F}_B \tag{4}$$

inducing a transverse Hall voltage of magnitude

$$V_H = Ew = \frac{IB}{tne} \tag{5}$$

where t is the thickness of the conductor in the $\hat{z}$ direction and w is the width in the $\hat{y}$ direction. Note that e carries the sign of the charge carrier, so measurements of the Hall voltage distinguish between positive and negative charge carriers.

III. LANDAU LEVELS

Under quantum mechanics, free electrons in a magnetic field no longer follow classical cyclotron orbits. Rather, the electrons are confined to a set of discrete orbits, known as Landau levels. Under a Landau gauge, a magnetic field in the $\hat{z}$ direction has the vector potential

$$\vec{A} = By\hat{x}, \tag{6}$$

and the resulting Hamiltonian is

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 \tag{7}$$

$$H = \frac{p_y^2}{2m} + \frac{1}{2m}\left(p_x - \frac{qBy}{c}\right)^2. \tag{8}$$

The x momentum p_x commutes with the Hamiltonian, so p_x can be replaced by its eigenvalues, $\hbar k_x$, where $k_x = \frac{2\pi n}{L_x}$ for some integer n, given a sample with width L_x. Also, $\frac{qB}{mc}$ is the classical cyclotron frequency ω_c. Then the Hamiltonian becomes

$$H = \frac{p_y^2}{2m} + \frac{m\omega_c^2}{2}\left(y - \frac{\hbar k_x}{m\omega_c}\right)^2, \tag{9}$$

which is a shifted harmonic oscillator centered at

$$y = \frac{\hbar k_x}{m\omega_c}. \tag{10}$$

This imposes another constraint on k_x, since the center of the harmonic oscillator must be inside the sample. Therefore,

$$0 \leq n \leq \frac{\omega_c m L_x L_y}{2\pi\hbar}, \tag{11}$$

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implying that there are

$$M = \left\lfloor \frac{\omega_c m A}{2\pi\hbar} \right\rfloor \tag{12}$$

states in each Landau level, with states in the m$^{\text{th}}$ level having energy $m\hbar\omega_c$. The eigenstates are described by $|m, k\rangle$, and the states in a given Landau level are strips of uniform probability that extend across the sample in the $\hat{x}$ direction and are localized in the $\hat{y}$ direction.

IV. QUANTUM HALL EFFECT

A. Appearance of Edge States

Consider a two-dimensional conductor that is long in the $\hat{x}$ direction and narrow in the $\hat{y}$ direction. A realistic sample will have a potential like that in Figure 1, with a higher energy near the edges. To first order, the energy of a Landau level state in this potential is

$$E = E_0 + \langle m, k|U(y)|m, k\rangle. \tag{13}$$

The $\hat{y}$ position of a state is a function of the momentum k as described by Equation 10. For large magnetic fields, an individual state takes only a small portion of the $\hat{y}$ extent of the sample, so the potential can be considered constant and the energy is simply shifted by $U(y)$. This causes a splitting of the degeneracy in Landau levels near the edge of the sample. The group velocity of electrons near the edge is given by

$$v_g = \frac{1}{\hbar}\frac{\partial E}{\partial k} = \frac{\partial U(y)}{\partial k} = \frac{\partial U(y)}{\partial y}\frac{\partial y}{\partial k} = \frac{\partial U(y)}{dy}\frac{c}{eB}. \tag{14}$$

Surprisingly, this indicates that the states moving in the $+\hat{x}$ direction are on the opposite edge as those moving in the $-\hat{x}$ direction, while electrons in the center have no net velocity.

B. Hall Voltage

Now, if a voltage V is applied in the $+\hat{x}$ direction, the Fermi velocity of the electrons moving in the $-\hat{x}$ direction will increase, while the Fermi velocity of the electrons in the $+\hat{x}$ direction will decrease, such that the difference between the Fermi energies is equal to the applied voltage. The spatial splitting of the edge states then means that the electrons on one side have a Fermi energy that is qV greater than the electrons on the other side. Thus, the Hall voltage V_H is equal to the applied voltage.

For momentum to be lost in the transport process, rightwards-moving electrons must scatter into leftwards-moving electrons or vice versa. However, if there are no Landau levels between the Fermi energies of the contacts, the mobile states at one edge cannot travel through the center of the sample to get to the other edge and scatter.

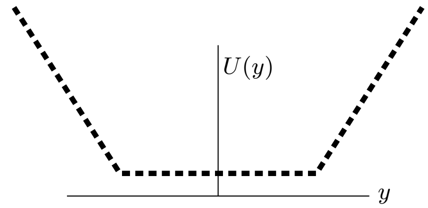

FIG. 1. The $\hat{y}$ potential of a narrow two-dimensional conductor pointing along the $\hat{x}$ direction

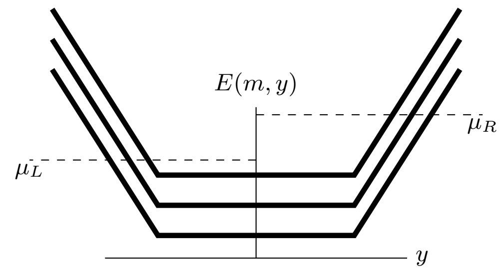

FIG. 2. The presence of an electric potential creates an asymmetry in the filling of the Landau levels

Thus, there is no momentum loss. This means that the voltage drop between two points on the same edge of the conductor is zero, so the longitudinal resistance of the sample is also zero.

Of course, when Landau levels are present between the Fermi energies of the contacts, particles can scatter through the center and lose momentum. This results in characteristic peaks in the magnetic field-resistance diagram in Figure 3.

C. Hall Current

The band structure for the conductor in a strong magnetic field is given in Figure 2. The current is equal to the single-band current times the number of bands, and is given by

$$I = \frac{4\pi e}{\hbar} M(\mu_R - \mu_L). \tag{15}$$

The Hall resistance takes the value V_H/I, which give the result

$$R_H = \frac{\hbar}{4\pi e^2 M}, \tag{16}$$

with M given in Equation 12. This causes a plateau in the Hall resistance for each Landau level as the magnetic field is varied (Figure 3).

V. QUANTUM ANOMALOUS HALL EFFECT

In a ferromagnetic material, a quantum anomalous Hall effect is present in addition to the standard effect

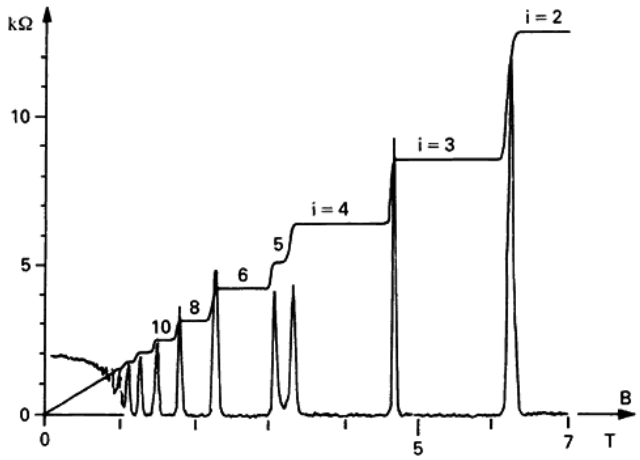

FIG. 3. The longitudinal resistance oscillates at high magnetic fields, while the Hall resistance experiences plateaus[2]

that depends not on the external magnetic field, but on the internal magnetization of the sample. Although this effect has been observed for nearly as long as the classical Hall effect, it is significantly less well understood than the classical or quantum effects. In the past few decades, a geometrical explanation based on the Berry phase has gained favor [3], but here we describe a recent theory based on coupling of magnetic moments to the angular momentum density of the electromagnetic field [4].

A. EM Angular Momentum - Magnetic Moment Coupling

First, we define two important quantities. The angular momentum density of an electromagnetic field is

$$\vec{\mathcal{J}} = \vec{r} \times \vec{p} = \frac{1}{c^2}\vec{r} \times (\vec{E} \times \vec{H}). \tag{17}$$

The magnetic toroidal moment is defined as

$$\vec{T} = \frac{1}{2}\int (\vec{r} \times \mathcal{M}) d^3 r \tag{18}$$

where $\mathcal{M}$ is the magnetization of the sample at a given point.

The toroidal moment couples to the cross product of the electric and magnetic fields [5] to give an energy of

$$U = a\vec{T} \cdot (\vec{E} \times \vec{H}), \tag{19}$$

where a is a constant that depends on the material. We can substitute in Equation 18 to find

$$U = a\frac{1}{2}\int d^3 r (\vec{r} \times \mathcal{M}) \cdot (\vec{E} \times \vec{H}) \tag{20}$$

$$= -a\frac{1}{2}\int (\vec{r} \times (\vec{E} \times \vec{H})) \cdot \mathcal{M} d^3 r = -\frac{ac^2}{2}\int \mathcal{J} \cdot \mathcal{M}\, d^3 r. \tag{21}$$

Hence, we observe a coupling between the electromagnetic angular momentum and magnetic moments.

B. Coupling in Ferromagnetic Materials

In a ferromagnet, the effective magnetic field $\vec{H}$ is a combination of an applied field and the internal field due to magnetization, given by

$$\vec{H} = (H_0 + \alpha\mathcal{M})\hat{z} = H\hat{z}. \tag{22}$$

where we consider a magnetic field only in the $\hat{z}$ direction. If we consider a single electron, we can eliminate the integral over space in the coupling energy equations. Furthermore, the magnetization in Equation 21 is simply the magnetic moment μ of the electron (but the magnetization in Equation 22 is the bulk magnetization). Converting to the Cartesian basis gives an energy of

$$U = -\frac{aH}{2}\left(z(E_x\mu_x + E_y\mu_y) - xE_x\mu_z - yE_y\mu_z\right). \tag{23}$$

Now the force on the electron due to the angular momentum coupling is given by

$$\begin{aligned} F = -\nabla_r U = \\ \frac{aH}{2}\left(-E_x\mu_z\hat{x} - E_y\mu_z\hat{y} + (E_x\mu_x + E_y\mu_y)\hat{z}\right). \end{aligned} \tag{24}$$

Note that this is separate from the Lorentz force.

C. Effect of Coupling on Transport

Applying the coupling force and the Lorentz force to the Drude model of electron transport gives

$$\begin{aligned} \frac{dp}{dt} &= -e\vec{E} - \frac{eH}{cm}\vec{p} \times \hat{z} \\ &+ \frac{aH}{2}(-E_x\mu_z\hat{x} - E_y\mu_z\hat{y} \\ &+ (E_x\mu_x + E_y\mu_y)\hat{z}) - \frac{\vec{p}}{\tau} \end{aligned} \tag{25}$$

where τ is the characteristic collision time of the system. In the steady state, the momenta are constant, so $dp/dt = 0$, and the Drude model gives in each direction

$$0 = -eE_x - \frac{eH}{cm}p_y - \frac{aH}{2}E_x\mu_z - \frac{p_x}{\tau} \tag{26}$$

$$0 = -eE_y + \frac{eH}{cm}p_x - \frac{aH}{2}E_y\mu_z - \frac{p_y}{\tau} \tag{27}$$

$$0 = -eE_x + \frac{aH}{2}(E_x\mu_x + E_y\mu_y) - \frac{p_z}{\tau} \tag{28}$$

As before, the current density j is nep/m. The steady-state equations can be summed over all electrons, replacing μs with $\mathcal{M}$s, giving

$$j_x = -\frac{ne^2\tau}{m}E_x - \frac{e\tau H}{cm}j_y - \frac{aHe\tau}{2m}E_x\mathcal{M}_z \quad (29)$$

$$j_y = -\frac{ne^2\tau}{m}E_y + \frac{e\tau H}{cm}j_x - \frac{aHe\tau}{2m}E_y\mathcal{M}_z \quad (30)$$

$$j_z = -\frac{ne^2\tau}{m}E_x + \frac{naHe\tau}{2m}(E_x\mathcal{M}_x + E_y\mathcal{M}_y). \quad (31)$$

Of course, in the Hall effect experimental setup, j_y and j_z are set to zero, and the magnetization is purely in the $\hat{z}$ direction. Therefore,

$$j_x = \left(-\frac{ne^2\tau}{m} - \frac{aHe\tau}{2m}\mathcal{M}_z\right)E_x \quad (32)$$

$$j_x = \left(\frac{nce}{H} + \frac{ac}{2}\mathcal{M}_z\right)E_y \quad (33)$$

$$E_z = 0. \quad (34)$$

The first line gives the longitudinal conductivity, and the second line gives the Hall conductivity. The anomalous Hall conductivity is the second term in the second equation, which depends on the magnetization of the sample but not on the external field.

VI. CONCLUSION

We have examined the very well-understood classical Hall effect, then seen how different environments can lead to striking results and interesting physics. Operating at very high magnetic fields results in a quantization of the Hall resistance; in fact, the quantization is so precise that it is used as the standard for the Ohm. We have further described a zero-field effect in ferromagnetic materials that enhances the Hall effect at non-zero fields.

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